数理解析研究所講究録
193 巻 1973 年 131-144



# A Generalization of a Prime Number Thorem of Rodosskii-Tatsuzawa for an algebraic Field

Teluhiko Hilano ( University of Tokyo )

## 1. Introduction

The Prime Number Theorem in an Arithmetic Progressions is stated as follows ;

Theorem A . If ( q , l ) = 1 and $q \leqq \exp ( c \sqrt{\log x} )$ then

$$\pi(x, q, l) = \frac{1}{\varphi(q)} \int_2^x \frac{dt}{\log t} + O\left(\frac{x^{\beta_1}}{\varphi(q)\log x}\right) + O\left(x e^{-c'\sqrt{\log x}}\right)$$

where $\beta_1$ is a so-called Siegel zero.

For q , which does not satisfy the above restriction , we have the following theorem ;

Theorem B ( Rodosskii - Tatsuzawa ) ( See , Prachar ( 8 ) Chap . 9 Satz 2 . 2 )

There exist constant $c_1$ and $c_2$ , such that , if

$$c_1 \log q \log\log q \leqq \log x \leqq c_2 ( \log q )^2$$

$$\pi(x, q, l) = \frac{1}{\varphi(q)} \int_2^x \frac{dt}{\log t} + O\left(\frac{x^{\beta_1}}{\varphi(q)\log x}\right) + O\left(\frac{x}{\varphi(q)\log x} e^{-c\frac{\log x}{\log q}}\right)$$

The prime ideal theorem in an ideal classes of an algebraic field , which is a generalization of theorem A , has been already obtained under the similar restriction.(see, Mitsui (5) )

The purpose of this paper is to offer a generalization of theorem B for an algebraic field under the similar condition. We now write it down ;

Theorem K is an algebraic field of degree n and $\mathfrak{f}$ is an integral ideal in K. We denote a product of $\mathfrak{f}$ and some infinite prime ideals by $\tilde{\mathfrak{f}}$ $h(\tilde{\mathfrak{f}})$ is the number of ideal classes mod $\tilde{\mathfrak{f}}$ and $\mathfrak{l}$ is one of the ideal classes . Then we have

for all $c > 0$ , there exist $c_1$ and $c_2$ such that, under the condition

$$c_1 \log N(\mathfrak{f}) \log\log N(\mathfrak{f}) \leqq \log x \leqq c(\log N(\mathfrak{f}))^2$$

$$\pi(x, \mathfrak{l}) = \sum_{\substack{N(\mathfrak{z})<x \\ \mathfrak{z}\in\mathfrak{l}}} 1 = \frac{1}{h(\tilde{\mathfrak{f}})}\int_2^x \frac{dt}{\log t} + O\left(\frac{x^{\beta_1}}{h(\tilde{\mathfrak{f}})\log x}\right) + O\left(\frac{x}{h(\tilde{\mathfrak{f}})\log x} e^{-c_2\frac{\log x}{\log N(\mathfrak{f})}}\right)$$

( The constant in O depend only on K and c . )

The auther has already given another proof of theorem B in ( 2 ) , using Gallagher's mean value theorem ( see, Gallagher ( 1 ) Theorem 2 ) In this paper, we use its generalization for an algebraic field , again . The functions $\mu(\mathfrak{a})$, $d(\mathfrak{a})$, $\Lambda(\mathfrak{a})$ and $\varphi(\mathfrak{a})$ are that of generalization for an algebraic field of Möbius , divisor , von Mangoldt and Euler function , respectively . All constants of all estimations depend only on K and c .

The auther wishes to express his thanks to Prof. T . Tatszawa , who taught him Theorem 2 and 3 in this paper , and encouraged him during preparing this paper .

2. The number of ideals of an ideal class mod $\tilde{\mathfrak{f}}$ whose norm lie in a given interval

Let K be an algebraic field of degree n with $r_1$ real conjugates and 2 $r_2$ complex conjugates and $\mathfrak{f}$ an integral ideal of K . And we put $\tilde{\mathfrak{f}} = \mathfrak{f}\mathfrak{z}_\infty^{(1)} \cdots \mathfrak{z}_\infty^{(\rho)}$ and h ($\tilde{\mathfrak{f}}$) is the number of ideal classes mod $\tilde{\mathfrak{f}}$ .Then ,

Theorem 1

$$h(\tilde{\mathfrak{f}}) = \frac{h\,\varphi(\tilde{\mathfrak{f}})}{e(\tilde{\mathfrak{f}})} \tag{1}$$

where h is the ideal class number of K , e ($\tilde{\mathfrak{f}}$) the number of residue classes mod $\tilde{\mathfrak{f}}$ of units of K , and $\varphi(\tilde{\mathfrak{f}}) = 2^{\rho}\varphi(\mathfrak{f})$

The proof of this theorem, for example , appears in Suetsuna ( 9) Chap . 2 Th . 4 ( p . 55 )

Cor .

$$h(\tilde{\mathfrak{f}}) \ll N(\mathfrak{f}) \tag{2}$$

Theorem 2

$$\sum_{\substack{N(\mathfrak{a})<x \\ \mathfrak{a}\in\mathfrak{l}}} 1 = \frac{1}{h(\tilde{\mathfrak{f}})}\frac{\varphi(\mathfrak{f})}{N(\mathfrak{f})} c_K x + O\left(\frac{1}{h(\tilde{\mathfrak{f}})}\frac{\varphi(\mathfrak{f})}{N(\mathfrak{f})} N(\mathfrak{f})^{\frac{1}{n}} x^{1-\frac{1}{n}}\right) \tag{3}$$

where $c_K$ is the constant only determined by K .

proof ) see, Tatsuzawa ( 10)

*Theorem* 3 If $y \ll x$, then

$$\sum_{\substack{x \leqq N(\mathfrak{a}) \leqq x+y \\ \mathfrak{a} \in \mathfrak{l}}} 1 \ll \frac{y}{h(\tilde{\mathfrak{f}})} + \frac{N(\mathfrak{f})^{\frac{1}{n}}}{h(\tilde{\mathfrak{f}})} x^{1-\frac{1}{n}} + 1 \quad (4)$$

We now choose a ideal class $\mathfrak{l}$ mod $\mathfrak{f}$, and fix it .

3 . A Generalization of Gallagher's Mean Value Theorem

*Lemma* If $z \ll y$, and $a(\mathfrak{a})$ are complex numbers , then

$$\sum_{\chi \bmod \tilde{\mathfrak{f}}} \Big| \sum_{y \leqq N(\mathfrak{a}) \leqq y+z} a(\mathfrak{a}) \chi(\mathfrak{a}) \Big|^2 \ll (z + N(\mathfrak{f})^{\frac{1}{n}} y^{1-\frac{1}{n}} + h(\tilde{\mathfrak{f}})) \sum_{y \leqq N(\mathfrak{a}) \leqq y+z} |a(\mathfrak{a})|^2 \quad (5)$$

proof ) By the orthogonal relation , Schwarz inequality and theorem 3 ,

we have
$$\sum_{\chi \bmod \tilde{\mathfrak{f}}} \Big| \sum_{y \leqq N(\mathfrak{a}) \leqq y+z} a(\mathfrak{a}) \chi(\mathfrak{a}) \Big|^2 \ll \sum_{\mathfrak{l} \bmod \tilde{\mathfrak{f}}} h(\tilde{\mathfrak{f}}) \Big| \sum_{\substack{y \leqq N(\mathfrak{a}) \leqq y+z \\ \mathfrak{a} \in \mathfrak{l}}} a(\mathfrak{a}) \Big|^2$$

$$\ll \sum_{\mathfrak{l} \bmod \tilde{\mathfrak{f}}} h(\tilde{\mathfrak{f}}) \Big( \sum_{\substack{y \leqq N(\mathfrak{a}) \leqq y+z \\ \mathfrak{a} \in \mathfrak{l}}} 1 \Big) \sum_{\substack{y \leqq N(\mathfrak{a}) \leqq y+z \\ \mathfrak{a} \in \mathfrak{l}}} |a(\mathfrak{a})|^2$$

$$\ll (z + N(\mathfrak{f})^{\frac{1}{n}} x^{1-\frac{1}{n}} + h(\tilde{\mathfrak{f}})) \sum_{y \leqq N(\mathfrak{a}) \leqq y+z} |a(\mathfrak{a})|^2$$

*Theorem* 4 . $\chi$ is a character mod $\tilde{\mathfrak{f}}$, and

$$S(\chi, t) = \sum a(\mathfrak{a}) \chi(\mathfrak{a}) N(\mathfrak{a})^{-it}$$

is an absolute convergent Dirichlet series. Then, if $T \geqq 1$ , we have

$$\sum_{\chi \bmod \tilde{\mathfrak{f}}} \int_{-T}^{T} |S(\chi, t)|^2 dt \ll \sum \big( h(\tilde{\mathfrak{f}}) T + N(\mathfrak{f})^{\frac{1}{n}} T N(\mathfrak{a})^{1-\frac{1}{n}} + N(\mathfrak{a}) \big) |a(\mathfrak{a})|^2 \quad (6)$$

proof ) We first refer the following inequality .

$S(t) = \sum a_n n^{it}$ is an absolute convergent Dirichlet series.
and $\tau = e^{1/T}$. Then

$$\int_{-T}^{T} |S(t)|^2 dt \ll T^2 \int_0^\infty \Big| \sum_{y}^{y\tau} a_n \Big|^2 \frac{dy}{y}$$

( See Gallagher (1) Theorem 1 ) Then we have

$$\sum_{\chi \bmod \mathfrak{f}} \int_{-T}^{T} |S(\chi,t)|^2 dt \ll T^2 \int_0^{\infty} \sum_{\chi \bmod \mathfrak{f}} \Big| \sum_{y \leq N(\mathfrak{a}) \leq y\tau} a_{\mathfrak{a}} \chi(\mathfrak{a}) \Big|^2 \frac{dy}{y}$$

By the assumption $T \geqq 1$ , we can apply the above lemma to the inner sum .

$$\ll T^2 \int_0^{\infty} \left( y(\tau-1) + N(\mathfrak{f})^{\frac{1}{n}} y^{1-\frac{1}{n}} + h(\tilde{\mathfrak{f}}) \right) \sum_{y \leq N(\mathfrak{a}) \leq y\tau} |a_{\mathfrak{a}}|^2 \frac{dy}{y}$$

The coefficient of $|a_{\mathfrak{a}}|^2$ is ( $n > 1$ )

$$T^2 \int_{N(\mathfrak{a})/\tau}^{N(\mathfrak{a})} \left( y(\tau-1) + N(\mathfrak{f})^{\frac{1}{n}} y^{1-\frac{1}{n}} + h(\tilde{\mathfrak{f}}) \right) \frac{dy}{y}$$

$$= h(\tilde{\mathfrak{f}})T + T^2 N(\mathfrak{a})(\tau-1)(1-\tau^{-1})$$
$$+ \frac{1}{1-\frac{1}{n}} N(\mathfrak{f})^{\frac{1}{n}} N(\mathfrak{a})^{1-\frac{1}{n}} (1-\tau^{-1+\frac{1}{n}}) T^2$$

$$\ll h(\tilde{\mathfrak{f}})T + N(\mathfrak{a}) + N(\mathfrak{f})^{\frac{1}{n}} T\, N(\mathfrak{a})^{1-\frac{1}{n}}$$

In the case $n = 1$ , the proof is similarly done . Then we get the theorem .

## 4 . Some properties on $L(s, \chi)$

In this paragraph we pick up some properties of $L(s, \chi)$ . These proofs can be found in Mitsui ( 5 ) or be similarly done as the case of Dirichlet L-functions , using theorem 2 and 3 .

Theorem 5 . The following estimations are true uniformly on $\sigma \geqq 1-\frac{1}{2n}$

$$L(s,\chi) = \sum_{N(\mathfrak{a})<M} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s} + E(\chi)\frac{N(\mathfrak{f})}{h(\mathfrak{f})\varphi(\mathfrak{f})} c_K \frac{M^{1-s}}{s-1}$$

$$+ O\left( |s| \frac{N(\mathfrak{f})^{\frac{1}{n}}}{M^{\sigma+\frac{1}{n}-1}} \right) \qquad (7)$$

*where* $E(\chi)$ *is 1 if* $\chi$ *is principal , 0 if not .*

Theorem 6

$$L(s,\chi)\,(s-1)^{E(\chi)} \ll N(\mathfrak{f})^2 (|t|+2)^{\frac{2}{3}n(1-\sigma)+E(\chi)} \log((|t|+2)N(\mathfrak{f}))$$

$$(\text{ uniformly } \ -\tfrac{1}{2} \leqq \sigma \leqq 4) \qquad (8)$$

Theorem 7

$$\frac{L'}{L}(s,\chi) = \sum_{|\gamma_0 - t| \leqq 1} \frac{1}{s-\rho_0} - \frac{E(\chi)}{s-1} + O(\log(N(\mathfrak{f})(|t|+2))) \qquad (9)$$

*where* $\rho_0 = \beta_0 + i\gamma_0$ *is a zero of* $L(s,\chi)$

Theorem 8 *There exists a constant* $A > 0$ *, such that , in the region*

$$\sigma \geqq 1 - \frac{A}{\log(N(\mathfrak{f})(|t|+2))}, \quad |t|>0 \qquad (10)$$

*all* $L(s,\chi)$ *whose character are defined mod* $\mathfrak{f}$ *, are zero-free .*

*For* $t = 0$ *, all* $L(s,\chi)$ *except one* $L(s,\chi_1)$ *where* $\chi_1$ *is a real character mod* $\mathfrak{f}$*, are zero-free in the interval* $1 \geqq \sigma \geqq 1 - \frac{A}{\log N(\mathfrak{f})}$

*The number of the exceptional zeros is at most one with multiplicity .*

## 5 . Notations and Remarks

*Now we put the restriction*

$$c_1 \log N(\mathfrak{f}) \log\log N(\mathfrak{f}) \leqq \log x \leqq c_2 (\log N(\mathfrak{f}))^2 \qquad (11)$$

and N ( $\mathfrak{f}$ ) is sufficiently large . We define the following functions .

$$\psi(x,\chi) = \sum_{N(\mathfrak{a})<x} \chi(\mathfrak{a}) \Lambda(\mathfrak{a}) \qquad (12)$$

$$\psi_1(x,\chi) = \sum_{N(\mathfrak{a})<x} \chi(\mathfrak{a}) \Lambda(\mathfrak{a}) \log\frac{x}{N(\mathfrak{a})} \qquad (13)$$

$$Q(s,\chi) = \sum_{N(\mathfrak{a})<N(\mathfrak{f})T^n} \frac{\mu(\mathfrak{a})\chi(\mathfrak{a})}{N(\mathfrak{a})^s} \qquad (14)$$

Furthermore , we assume that $\log T \ll \log N(\mathfrak{f})$ (15)

$$\alpha = 1+\frac{1}{\log x}, \quad \beta = 1-\frac{1}{2n}+\frac{1}{\log x} \qquad (16)$$

$$\psi(x,\mathfrak{l}) = \sum_{\substack{N(\mathfrak{a})<x \\ \mathfrak{a}\in\mathfrak{l}}} \Lambda(\mathfrak{a}) \qquad (17)$$

and

$$\psi_1(x,\mathfrak{l}) = \sum_{\substack{N(\mathfrak{a})<x \\ \mathfrak{a}\in\mathfrak{l}}} \Lambda(\mathfrak{a}) \log\frac{x}{N(\mathfrak{a})} \qquad (18)$$

Then ,

$$\psi_1(x,\mathfrak{l}) = \frac{1}{h(\tilde{\mathfrak{f}})} \sum_{\chi \bmod \mathfrak{f}} \overline{\chi(\mathfrak{l})}\, \psi_1(x,\chi) \qquad (19)$$

We begin with the estimation of $\psi_1(x,\mathfrak{l})$

6 . The estimation of $\psi_1(x,\mathfrak{l})$

6 . 1 Transformation of $\psi_1(x,\mathfrak{l})$

<u>Lemma</u> $a > 0$ , then

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{x^s}{s^2}\,ds = \begin{cases} 0 & 0<x\leqq 1 \\ \log x & x\geqq 1 \end{cases} \qquad (20)$$

The proof of this lemma is immediately done .

By the above lemma and the definition of $L(s, \chi)$ , we have

$$\psi_1(x, \chi) = -\frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} \frac{L'}{L}(s, \chi)\frac{x^s}{s^2}\,ds$$

$$= -\frac{1}{2\pi i}\left(\int_{\alpha - iT}^{\alpha + iT} + \int_{\alpha - i\infty}^{\alpha - iT} + \int_{\alpha + iT}^{\alpha + i\infty}\right)\frac{L'}{L}(s, \chi)\frac{x^s}{s^2}\,ds$$

$$= -\frac{1}{2\pi i}\left(I_1(x) + I_2(x) + I_3(x)\right) \qquad (21)$$

( Say ! )

By theorem 8 and the condition ( 15) , we may assume the following conditions .

1 ) All $L(s, \chi)$ except one real zero of $L(s, \chi_1)$ are zero- free in the region

$$\sigma \geqq 1 - \frac{A}{\log N(f)}, \quad |t| \leqq T \qquad (A > 0) \qquad (22)$$

2 ) All zeros and poles of all $L(s, \chi)$ mod $f$ are far from the line

$$\sigma = 1 - \frac{A}{\log N(f)}, \quad |t| \leqq T \qquad (23)$$

by $\gg \dfrac{1}{\log N(f)}$ .

We put $\gamma = 1 - \dfrac{A}{\log N(f)}$ (24)

We change the way of integration ,and have

$$I_1(x) = -2\pi i\, E(\chi)\, x + 2\pi i\, E_1(\chi)\frac{x^{\beta_1}}{\beta_1^2} + \int_{\gamma - iT}^{\gamma + iT}\frac{L'}{L}\frac{x^s}{s^2}\,ds$$
$$+ \left(\int_{\gamma + iT}^{\alpha + iT} - \int_{\gamma - iT}^{\alpha - iT}\right)\frac{L'}{L}(s, \chi)\frac{x^s}{s^2}\,ds$$

$$= -2\pi i\, E(\chi)\, x + 2\pi i\, E_1(\chi)\frac{x^{\beta_1}}{\beta_1^2} + I_4(x) + I_5(x) \qquad (25)$$

(Say!)

$$E_1(\chi) = \begin{cases} 1 & \text{if there is a zero in the region} \quad . \\ 0 & \text{otherwise} \quad ) \end{cases} \tag{26}$$

We put $E_1^* = \sum_{\chi \bmod f} E_1(\chi)\overline{\chi(l)}$ (27)

Using ( 14 ) , we transform $I_4(\chi)$ as follows ;

$$I_4(\chi) = \int_{\gamma - iT}^{\gamma + iT} \frac{L'}{L}(s,\chi)(1 - L(s,\chi)Q(s,\chi))^2 \frac{x^s}{s^2} ds$$

$$+ \left( \int_{\beta - iT}^{\beta + iT} + \int_{\beta + iT}^{\gamma + iT} - \int_{\beta - iT}^{\gamma - iT} \right)(2L'Q - L'LQ^2)\frac{x^s}{s^2} ds$$

$$= I_7(\chi) + I_8(\chi) + I_9(\chi) + I_{10}(\chi) \quad (\text{say}) \tag{28}$$

6 . 2 Estimation of $I_2(\chi)$ and $I_3(\chi)$

By theorem 7 , we have the following estimations on the line $\sigma = \alpha$

$$\frac{L'}{L}(s,\chi) \ll \log(N(f)(|t|+2)) \log x \tag{29}$$

Hence ,

$$I_3(\chi) + I_4(\chi) \ll x \log x \int_T^{\infty} \frac{\log N(f)(|t|+2)}{\alpha^2 + T^2} dt \ll \frac{x \log^2 x}{T} \tag{30}$$

6 . 3 Estimation of $I_5(\chi)$ and $I_6(\chi)$

As the above paragraph , we have

$$I_5(\chi) + I_6(\chi) \ll \frac{\log^2 x}{T^2} \int_\gamma^\alpha x^\sigma d\sigma \ll \frac{x \log^2 x}{T^2} \tag{31}$$

6 . 4 Estimation of $I_9(\chi)$ and $I_{10}(\chi)$

By using Cauchy's integration formula , we have

$$L'(s,\chi) \ll N(f)^2 (|t|+2)^{\frac{2}{3}n(1-\sigma)} \log^3 N(f)(|t|+2) \tag{32}$$

*On the other hand , the following estimations are obtained by using convexity argument .*

$$Q(s,\chi) \ll N(f)\, T^{n(1-\sigma)} \log x$$
$$\text{for} \quad \beta \leq \sigma \leq \alpha . \tag{33}$$

*Therefore we have*

$$I_9(x) + I_{10}(x) \ll N(f)^6 \frac{x}{T^2} \log^5 x \int_\beta^\gamma \left(\frac{T^{\frac{10}{3}n}}{x}\right)^{1-\sigma} d\sigma$$
$$\ll N(f)^6 \frac{x^\gamma}{T^2} \log^5 x \tag{34}$$
$$\left(\text{if} \quad T^{\frac{10}{3}n} \leq x \right)$$

*6 . 5 Estimation of $I_8(\chi)$*

*By theorem 6 , ( 32) ,and ( 33) , we have*

$$I_8(x) \ll N(f)^6\, T^{\frac{10}{3}n(1-\beta)} x^\beta \log^6 x$$
$$= N(f)^6\, T^{5/3} x^\beta \log^6 x \tag{35}$$

*6 . 6 Estimations of $I_7(\chi)$*

*Putting $M = N(f)\,T^n$ in theorem 5 , we have on the line* $\sigma = \alpha$

$$L(s,\chi)Q(s,\chi) - 1 = \sum_{N(f)T^n \leq N(\mathfrak{r}) \leq (N(f)T^n)^2} b(\mathfrak{r})\chi(\mathfrak{r})N(\mathfrak{r})^{-s} + O\left(|Q(s,\chi)|\left(\frac{|s|}{T} + E(x)\log x\right)\right) \tag{36}$$

*where* $$b(\mathfrak{r}) = \sum_{\mathfrak{z}|\mathfrak{r},\, N(\mathfrak{z})<N(f)T^n} \mu(\mathfrak{z}) \tag{37}$$

*Therefore ,*

$$\sum_{\chi \bmod \tilde{f}} |I_\gamma(x)| \ll x^\gamma \log^2 x \left\{ \int_{-T}^{T} x \frac{\sum_{\bmod \tilde{f}} \left| \sum_{N(f)T^n \leq N(\mathfrak{r}) \leq (N(f)T^n)^2} b(\mathfrak{r}) \chi(\mathfrak{r}) N(\mathfrak{r})^{-\gamma - it} \right|^2}{|\gamma + it|^2} dt \right.$$

$$\left. + \frac{1}{T^2} \int_{-T}^{T} \sum_{\chi \bmod \tilde{f}} |Q(\gamma + it, \chi)|^2 dt + \log^2 x \int_{-T}^{T} \frac{|Q(\gamma + it, \chi_0)|^2}{|\gamma + it|^2} dt \right\} \qquad (38)$$

*We now apply Gallagher's mean value theorem to the above integrals .*

*Then we have*

$$\sum_{\chi \bmod \tilde{f}} \int_{-T}^{T} |Q(\gamma + it, \chi)|^2 dt \ll \sum_{N(\mathfrak{r}) < N(f)T^n} \left( h(\tilde{f})T + N(f)^{\frac{1}{n}} T N(\mathfrak{r})^{1-\frac{1}{n}} + N(\mathfrak{r}) \right) \frac{1}{N(\mathfrak{r})^{2\gamma}}$$

$$\ll N(f)T \qquad (39)$$

*and*

$$\sum_{\chi \bmod \tilde{f}} \int_{-T}^{T} \frac{\left| \sum_{N(f)T^n \leq N(\mathfrak{r}) \leq (N(\mathfrak{r})T^n)^2} b(\mathfrak{r}) \chi(\mathfrak{r}) N(\mathfrak{r})^{-\gamma - it} \right|^2}{|\gamma + it|^2} dt$$

$$\ll \sum_{N(f)T^n \leq N(\mathfrak{r}) \leq (N(f)T^n)^2} \left( h(\tilde{f})T + N(f)^{\frac{1}{n}} T N(\mathfrak{r})^{1-\frac{1}{n}} + N(\mathfrak{r}) \right) \frac{|b(\mathfrak{r})|^2}{N(\mathfrak{r})^{2\gamma}}$$

$$\ll \log^4 x \qquad (40)$$

*(We use* $\sum_{N(\mathfrak{r}) < x} |b(\mathfrak{r})|^2 \ll \sum_{N(\mathfrak{r}) \leq x} |d(\mathfrak{r})|^2 \ll x \log^3 x$ )

*The last integral is trivially estimated and we obtain*

$$\sum_{\chi \bmod \tilde{f}} I_\gamma(x) \ll x^\gamma \left( \log^4 x + \frac{N(f)}{T} \right) \qquad (41)$$

*6 . 7 Estimation of* $\psi(x, l)$

*Now we collect the results of preceding paragraphs , then we have*

$$\psi_1(x, l) = \frac{x}{h(\tilde{f})} - E_1^* \frac{x^{\beta_1}}{h(\tilde{f}) \beta_1^2} + O\left( \frac{x}{T} \log^2 x \right)$$

$$+ O\left( x^{1-\frac{1}{2n}} N(f)^6 T^{\frac{5}{3}} \log^6 x \right) + O\left( \frac{x^6}{h(\tilde{f})} \log^4 x \right) \quad (42)$$

*Taking* $T = N(f)^4$, *we have*

$$\frac{x}{T} \log^2 x \ll \frac{x}{h(\tilde{f})} e^{-c_3 \frac{\log x}{\log N(f)}} \quad (43)$$

$$x^{1-\frac{1}{2n}} N(f)^6 T^{\frac{5}{3}} \log^5 x \ll \frac{x}{h(\tilde{f})} N(f)^{-c_4 \log\log N(f)}$$
$$\ll \frac{x}{h(\tilde{f})} e^{-c_5 \frac{\log x}{\log N(f)}} \quad (44)$$

*and*

$$\frac{x^{\gamma}}{h(\tilde{f})} \log^4 x \ll \frac{x}{h(\tilde{f})} e^{-\frac{A}{2} \frac{\log x}{\log N(f)}} \quad (45)$$

*Therefore we get*

$$\psi_1(x, \chi) = \frac{x}{h(\tilde{f})} - E_1^* \frac{x^{\beta_1}}{h(\tilde{f})\beta_1^2} + O\left( \frac{x}{h(\tilde{f})} e^{-c_6 \frac{\log x}{\log N(f)}} \right) \quad (46)$$

*7. Estimation of* $\psi(x, \chi)$

*Proposition 1. For all* $c^* > 0$, *there exists a constant* $c_1^*$ $0$,

*such that, under the condition*

$$c_1^* \log N(f) \log\log N(f) \leq \log x \leq c^* (\log N(f))^2$$

$$\psi(x, \chi) = \frac{x}{h(\tilde{f})} - E_1^* \frac{x^{\beta_1}}{h(\tilde{f})\beta_1} + O\left( \frac{x}{h(\tilde{f})} e^{-c_2^* \frac{\log x}{\log N(f)}} \right) \quad (47)$$

*proof) Let* $c_2 = 2c^*$ *and take* $\theta$, *such that* $0 \leqq \theta \leqq 1$,

*we have by (46),*

$$\psi_1(x+\theta x, \chi) - \psi_1(x, \chi) = \frac{\theta x}{h(\tilde{f})} - E_1^* \frac{x^{\beta_1}}{h(\tilde{f})\beta_1} (\theta + O(\theta^2))$$
$$+ O\left( \frac{x}{h(\tilde{f})} e^{-c_8 \frac{\log x}{\log N(f)}} \right) \quad (48)$$

*And by the definition of* $\psi_1(x, \chi)$, *we get*

$$\psi_1(x+\theta x, l) - \psi_1(x, l) = \sum_{N(\alpha)<x,\ \alpha\in l} \Lambda(\alpha)\log(1+\theta) + \sum_{\substack{x\le N(\alpha)<x+\theta x\\ \alpha\in l}} \Lambda(\alpha)\log\frac{x}{N(\alpha)} \qquad (49)$$

*Using theorem 3 , we obtain*

$$\sum_{\substack{x\le N(\alpha)<x+\theta x\\ \alpha\in l}} \Lambda(\alpha)\log\frac{x}{N(\alpha)} = O\left(\theta\log x\left(\frac{\theta x}{h(f)} + \frac{N(f)^{\frac{1}{n}}}{h(f)}x^{1-\frac{1}{n}} + 1\right)\right) \qquad (50)$$

*and*

$$\sum_{N(\alpha)\le x,\ \alpha\in l} \Lambda(\alpha)\log(1+\theta) = \theta\psi(x, l) + O\left(\frac{\theta^2 x\log x}{h(f)}\right) \qquad (51)$$

*The proposition is immediately deduced , taking*

$$\theta = e^{-\frac{c_6}{2}\frac{\log x}{\log N(f)}} \qquad (52)$$

*The theorem is easily proved from this proposition .*

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